Solutions to Seminar exercise 4

1.Ohm's law for a line of a given length, type of metal and thickness

$$P_L = I^2 R = \frac{P_o^2}{V_o^2} R = \frac{P_o^2}{V_o^2}\frac{2L\rho}{A},$$

$$\frac{\partial P_L}{\partial P_o} = \frac{2P_o}{V_o^2}\frac{2L\rho}{A} > 0, \; \frac{\partial^2 P_L}{\partial P_o^2} = \frac{2}{V_o^2}\frac{2L\rho}{A} > 0$$

$$\frac{\partial P_L}{\partial V_o} = -\frac{2P_o^2}{V_o^3}\frac{2L\rho}{A} < 0$$

Notice that the second derivative is a constant.

The shadow prices:

Shadow price on the energy balance; change in objective function of relaxing the constraint $x_{jt} + e_{jt}^L = e_{jt}^H$. (Use a positive constant instead of 0, then differentiate w.r.t. this constant and check the marginal value function (value of the objective functin at optimal solution) when the constant goes toward 0.)

Shadow price on the line capacity:

Standard intrpretation, use the envelope theorem and differentiate w.r.t. $\bar{x}_j$

2a.The Lagrangian

$$L = \sum_{t=1}^{2} \int_{z=0}^{\sum_{j=1}^{2} x_{jt}} p_t(z)dz$$

$$-\sum_{t=1}^{2}\sum_{j=1}^{2}\lambda_{jt}(R_{jt} - R_{j,t-1} - w_{jt} + e_{jt}^H)$$

$$-\sum_{t=1}^{2}\sum_{j=1}^{2}\gamma_{jt}(R_{jt} - \bar{R}_j)$$

$$-\sum_{t=1}^{2}\sum_{j=1}^{2}\tau_{jt}(x_{jt} + e_j^L(x_{jt}) - e_{jt}^H)$$

$$-\sum_{t=1}^{2}\sum_{j=1}^{2}\mu_{jt}(x_{jt} - \bar{x}_j)$$

Loss function assumed independent of time, in line with assuming the capacity of the line independent of time

First-order conditions

$$\frac{\partial L}{\partial x_{jt}} = p_t(x_t) - \tau_{jt} - \tau_{jt}\frac{\partial e_j^L}{\partial x_{jt}} - \mu_{jt} \le 0 \ (= 0 \text{ for } x_{jt} > 0)$$

$$\frac{\partial L}{\partial e_{jt}^H} = -\lambda_{jt} + \tau_{jt} \le 0 \ (= 0 \text{ for } e_{jt}^H > 0)$$

$$\frac{\partial L}{\partial R_{jt}} = -\lambda_{jt} + \lambda_{j,t+1} - \gamma_{jt} \le 0 \ (= 0 \text{ for } R_{jt} > 0)$$

$$\lambda_{jt} \ge 0 (= 0 \text{ for } R_{jt} < R_{j,t-1} + w_{jt} - e_{jt}^H)$$

$$\gamma_{jt} \ge 0 (= 0 \text{ for } R_{jt} < \bar{R}_j)$$

$$\mu_{jt} \ge 0 (= 0 \text{ for } x_{jt} < \bar{x}_j) \ , \ j = 1,2 \ , \ t = 1,2$$

2b. Decomposition of consumer price

$$p_t(x_t) = \lambda_j + \lambda_j \frac{\partial e_j^L}{\partial x_{jt}} + \mu_{jt} = \lambda_j(1 + \frac{\partial e_j^L}{\partial x_{jt}}) + \mu_{jt}, \ j = 1,2 \ , \ t = 1,2$$

The consumer price is consisting of three parts; water value, value of marginal loss, and shadow price on line capacity constraint.

The difference between the social consumer prices for the two time periods, assuming water value for each plant the same for the two periods, but differing between plants:

$$p_t(x_t) = \lambda_{jt} + \lambda_{jt}\frac{\partial e_j^L}{\partial x_{jt}} + \mu_{jt} \ , \ t = 1,2 \, , \ j = 1,2 \Rightarrow$$

$$p_2(x_2) - p_1(x_1) = \lambda_j(\frac{\partial e_j^L}{\partial x_{j2}} - \frac{\partial e_j^L}{\partial x_{j1}}) + (\mu_{j2} - \mu_{j1}), \ j = 1,2$$

The sum of loss-adjusted water values and shadow value of congestion must be equal for the plants for each time period.

$$\lambda_1(1 + \frac{\partial e_1^L}{\partial x_{1t}}) + \mu_{1t} = \lambda_2(1 + \frac{\partial e_2^L}{\partial x_{2t}}) + \mu_{2t}, \ t = 1,2$$

Let us assume no congestion on the lines for the two periods. To make a high-production period at least one plant must produce sufficiently more in period 2 because the consumer price is higher in period 2 by definition. But then both plants must produce more acccording to the first-order condition for total consumption, i.e. the difference between the marginal losses must be positive for both plants.

Marginal loss greater on line 1: this is most likely due to the longer distance if we assume the same voltage, etc. on the lines.

If water inflow to plant 1 is greater than inflow to plant 2, then the marginal loss for plant 1 will reasonably be greater for both periods and consequently the water value for plant 1 will be less than for plant 2 looking at the condition

$$\lambda_1(1+\frac{\partial e_1^L}{\partial x_{1t}}) = \lambda_2(1+\frac{\partial e_2^L}{\partial x_{2t}}),\ t=1,2$$

Differences in marginal loss values water values must have the same sign in both periods. The marginal loss increases more rapidly for plant 1 than plant 2 (L_1 greater than L_2 in Ohm's law), therefore plant 1 will use relatively more water in period 1 than plant 2.

2c. Nodal prices

Follow from the discussion of decomposition of consumer price. Water value is the price the producer gets in the whole-sale spot market

Implementation problem:

Nodal prices change continuously; what is the optimal/feasible time aggregation